SOLUTIONS OF PROBLEMS IN NO. 5.

Solutions of problems in No. 5 have been received as follows: From Prof. P. E. Chase, 24; S. J. Child, 21 & 22; Geo. M. Day, 20, 21 & 22; Prof. A. B. Evans, 20, 21, 22, 23 & 24; Henry Gunder, 20, 21, 22 & 23; Prof. E. W. Hyde, 20; H. Heaton, 24; Artemas Martin, 20, 21, 22 & 23; Miss Esther Matthews, 21, 22 & 23; L. E. Newcomb, 20, 21, 22 & 23; A. W. Phillips, 20, 21, 22 & 23; Henry A. Roland, 20, 21, 22, 23 & 24; L. Regan, 21 & 22; S. W. Salmon, 20, 21 & 23; Walter Sirerly, 20, 21, 22 & 23; T. P. Stowell, 20 & 22; Prof. D. M. Sensenig, 20 & 21; E. B. Seitz, 20, 21, 22 & 23; Prof. J. Scheffer, 20, 21, 22 & 23; Prof. H. T. J. Ludwick, 21 & 22.

We received elegant solutions of all the problems in No. 4, from Artemas Martin, but by an oversight failed to give him credit for them in No. 6.

20. "It is required to circumscribe about a given parabola an isosceles triangle whose area shall be a minimum."

SOLUTION BY ARTEMAS MARTIN, ERIE, PA.

Let a be the altitude of the parabola, and x and y the coordinates of the point of contact with the triangle; then $a + x$ is the altitude of the triangle, and we have by similar triangles

$$2\,x : 2\,y :: a + x : \frac{y\,(a + x)}{x},$$

the base of the triangle.

Its area $= \dfrac{y\,(a + x)^2}{2\,x} = \dfrac{(a + x)^2\sqrt{2\,p\,x}}{2\,x}$, since $y^2 = 2\,p\,x$.

$$\therefore \frac{(a + x)^2\sqrt{2\,p\,x}}{2\,x} = \text{minimum},$$

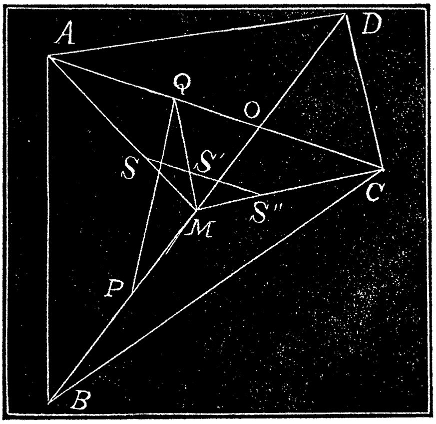
$$\text{or } \frac{(a + x)^4}{x} = \text{minimum} = u.$$

$$\frac{d\,u}{d\,x} = \frac{4\,x\,(a + x)^3 - (a + x)^4}{x^{\;2}} = 0.$$

$$\therefore a + x = 4\,x = \frac{4}{3}\,a, \text{ the altitude of the triangle.}$$

21. "$A\ B\ C\ D$ is a quadrilateral; O, the intersection of the diagonals; P, Q, points in BD, AC, such that $Q A = O C$ and $P B = O D$. Prove that the centre of gravity of the quadrilateral coincides with that of the triangle $O P Q$."

SOLUTION BY GEO. M. DAY, LOCKPORT, N. Y.



Let M be the middle point of $D B$ = middle of $P\ O$. Take $M S = \frac{1}{3} M\ A$ and draw $S\ S''$ parallel to $A\ C$; join $M\ Q$, then S, S', S'' are the centres of gravity of $A\ B\ D$, $O\ P\ Q$, $D\ B\ C$ respectively. We have $A\ B\ D : B\ D\ C :: A\ O : O\ C$, and $S'\ S'' : S\ S' :: A\ O : O\ C$. $\therefore A\ B\ D : B\ D\ C :: S'\ S'' : S\ S'$, hence S' is the centre of gravity of the quadrilateral $A\ B\ C\ D$.

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22. "Show that the distance from a vertex of any plane triangle to the point where the opposite escribed circle touches the sides meeting at the vertex is constant and equal to half the sum of the sides of the triangle."

SOLUTION BY E. B. SEITZ, GREENVILLE, O.

Let $A\ B\ C$ be any triangle, D, E the points at which the escribed circle opposite A touches $A\ B$, $A\ C$ produced. Now $B\ D + C\ E = B\ C$; $\therefore A\ D + A\ E = A\ B + A\ C + B\ C$. But $A\ D = A\ E$; $\therefore A\ D = \frac{1}{2}(A\ B + A\ C + B\ C)$.

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23. "If the brightness of the moon be equal to the brightness of the clouds by day, show that the light of an overcast day is to that of a full moon-lit night as $8\ (360)^2 : \pi^2$; the diameter of the moon being $30'$."

SOLUTION BY PROF. W. C. ESTY, AMHERST, MASS.

Let r = the radius of the cloud surface.

Then $2\ \pi\ r^2$ = the visible cloud area,

$$\frac{\pi r}{360} = \text{the apparent diameter of moon,}$$

$$\frac{\pi}{4}\left(\frac{\pi^2 r^2}{360^2}\right) = \frac{\pi^3 r^2}{4.(360)^2} = \text{the apparent area of the moon,}$$

or the portion of the cloud surface covered by its disk.

The ratio required is

$$2\pi r^2 : \frac{\pi^3 r^2}{4.(360)^2} = 8.\ (360)^2 : \pi^2.$$

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24. "There are m labels, to be distributed by lot among m different articles. Required the probable amount of coincidence in two independent allotments."

SOLUTION BY PROF. PLINY EARLE CHASE, HAVERFORD COLLEGE, PA.

Suppose the articles to be similarly arranged in the two allotments. Then, among the possible permutations of the labels, there is only one in which they will all coincide, and none in which $m - 1$ will coincide.

If all but two coincide, those two must change places. There are, $\therefore$ as many such arrangements as we can make selections of 2 out of m, or $\frac{m(m-1)}{2\,!}$.

If all but three coincide, those three must change places in such a way that neither will occupy its original place. This can be done in two ways with each group of 3; there are, $\therefore$, twice as many such arrangements as the number of selections that we can make of 3 out of m, or $2 \times \frac{m(m-1)(m-2)}{3\,!}$.

Tabulating and differencing those results, it will be seen that the number of arrangements in which there are n displacements out of m, is

$$\Delta^n 0\,! \frac{\times m \ldots (m-n+1)}{n\,!}.$$

The value of this expression can be easily found, for

$$\Delta^n 0\,! = \frac{n\,!}{0\,!} - \frac{n\,!}{1\,!} + \frac{n\,!}{2\,!} - \ldots$$

$$n\,\Delta^{n-1} 0\,! = \frac{n\,!}{0\,!} - \frac{n\,!}{1\,!} + \frac{n\,!}{2\,!} - \ldots (-)^{n+1}\,1. \quad \text{Hence, by subtraction,}$$

$$\Delta^n 0\,! = n\,\Delta^{n-1}\ 0\,! (-)^n\,1$$

There will $\therefore$ be m displacements, or 0 coincidences, in

$$\Delta^m 0 ! = \frac{m\,!}{e}^{*} = .36788\, m\,!$$

arrangements, leaving for the probability of one or more coincidences, .63212 m ! out of m ! possible arrangements,—which is a probability of about $\frac{5}{8}$.

There will also be $m - 1$ displacements, or 1 coincidence, in .36788 m ! arrangements. Deducting this amount from .63212 m ! there remains only a chance of $\frac{26434}{100000}$ for more than one coincidence. There would $\therefore$ probably be one accidental coincidence, and no more.

SOLUTION BY PROF. W. C. ESTY, AMHERST, MASS.

There are $\underline{|m}$ ways in which the labels may be arranged. If we take any particular set of labels, r in number, and designate by A_r the number of arrangements in which this set of labels all fail to coincide and then multiply this number A_r by the number of sets consisting of r labels each that may be formed out of m labels, we shall have the entire number of arrangements in which there are r labels that fail of coincidence. This product

$$= A_r \frac{\underline{|m}}{\underline{|r}\ \underline{|m-r}}.$$

It is evident that the same product gives the number of cases in which there are $m - r$ coincidences. Putting, in the above formula, for r every number in succession from $r = 0$, up to $r = m$ inclusive and taking the sum of the results we have all the arrangements possible $\underline{|m}$ in number *i. e.*

$$\underline{|m} = \sum_{o}^{m}{}_{r}.\ A_r \frac{\underline{|m}}{\underline{|r}\ \underline{|m-r}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

This is the equation given on page 336 of Todhunter's History of the Theory of Probabilities where a general formula for computing A_r is given, which is not needed for our present problem.

The probability that r labels will fail and that $m - r$ labels will not fail to coincide may be designated by P_r which can be found by the equation

*The student will understand that the notation m ! means the same as $|\,m$, and that each represents 1. 2. 3. . . . . m.—ED.

$$P_r = A_r \frac{\underline{|m}}{\underline{|r}\;\;\underline{|m-r}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

when A_r is known.

Let $E_r =$ the mathematical expectation of coincidence, or the *probability* of an arrangement containing any assigned number of coincidences as $m - r$, multiplied by the number of coincidences. Then will

$$E_r = (m-r)\,P_r \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Now the total expectation of coincidence will be a summation of all the terms E_r from $r = 0$ to $r = m - 1$ inclusive. Call this total expectation, or probable amount of coincidence, C. Then

$$C = \sum_{0}^{m-1}{}_{r}\, E_r = \sum_{0}^{m-1}{}_{r}\, (m-r)\,P_r = \sum_{0}^{m-1}{}_{r}\, \frac{(m-r)\,A_r}{\underline{|r}\;\;\underline{|m-r}}\ddagger$$

$$= \sum_{0}^{m-1}{}_{r}\, \frac{A_r}{\underline{|r}\;\;\underline{|m-1-r}} = \frac{1}{\underline{|m-1}} \sum_{0}^{m-1}{}_{r}\, \frac{A_r\,\underline{|m-1}}{\underline{|r}\;\;\underline{|m-1-r}} = \frac{\underline{|m-1}}{\underline{|m-1}} \text{ [by (1)]}$$

$$=1.$$

PROBLEMS.

30. By Otis Shepard, Goodland, Ind.—Prove that

$$\frac{\sqrt[4]{a} + \sqrt[4]{b}}{\sqrt[4]{a} - \sqrt[4]{b}} = \frac{a + b + 2\sqrt{ab} + 2\sqrt[4]{a^3 b} + 2\sqrt[4]{a b^3}}{a-b}.$$

31. By Prof. D. Kirkwood, Bloomington, Ind.—Solve the equation

$$x = \sqrt[4]{x + \frac{x}{\sqrt[4]{x + \frac{x}{\sqrt[4]{x + \&c}}}}}$$

and express the value of x in a finite number of terms.

32. By Prof. D. Trowbridge, Waterburgh, N. Y.—There are two spheres of equal size and of exactly the same appearance, one solid

‡In order that we may obtain the probable number of coincidences *in each independent allotment* the sum, C, must be divided by the number of permutations, $\underline{|m}$; $\therefore$

$$\frac{C}{\underline{|m}} = \frac{1}{\underline{|m}} \sum_{0}^{m-1}{}_{r}\, E_r = \frac{1}{\underline{|m}} \sum_{0}^{m-1}{}_{r}\, (m-r)\,P_r = \sum_{0}^{m-1}{}_{r}\, \frac{(m-r)\,A_r}{\underline{|r}\;\;\underline{|m-r}}.$$ —Ed.